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The instanton moduli spaces as algebraic sets

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The algebraic equations are derived quite explicitly which describe the framed instanton moduli space with the gauge group SU(2) and arbitrary instanton number as an algebraic set in a finite-dimensional complex space.

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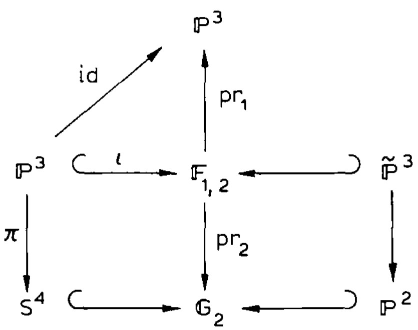
1. Introduction

As is well known, the Penrose transformation enables one to state a one-to-one correspondence between instanton solutions on S^4 and holomorphic vector bundles on $\mathbb{P}^3$ obeying some additional reality conditions [1]. This is the basis for the ADHM construction [2]. In this paper we are going to develop an alternative approach to instantons, which was proposed in ref. [3]. Compared with the ADHM construction, which is based on the monad description of holomorphic bundles, our approach has its origin in some ideas of Takasaki [4] about locally defined self-dual Yang–Mills fields. Giving these ideas a clear geometric interpretation in the global case one is able to relate to each gauge equivalence class of instanton solutions a rational matrix-valued function on $\mathbb{P}^3$ with some special properties. The advantage is that the gauge freedom is completely eliminated. This is in contrast to the ADHM construction, in the framework of which an action of a finite-dimensional Lie group still survives as a reminiscence of the infinite-dimensional group of gauge transformations.

Let $\pi:\mathbb{P}^3\rightarrow S^4$ be the Penrose projection. The fibres are projective lines in $\mathbb{P}^3$, the so-called real lines. Fix a real line $\mathscr{L}_0\subset\mathbb{P}^3$. Every instanton bundle is holomorphically trivial on all real lines. We shall consider framed holomorphic bundles. This means that the data include a choice of a holomorphic trivialization on $\mathscr{L}_0$. Let $M(r,c)$ designate the moduli space of framed instanton bundles on $\mathbb{P}^3$, where r is the rank of the vector bundles under consideration [hence the gauge group is $SU(r)$] and c is the value of the second Chern class.

Let $\mathbb{P}^2$ be a projective plane in $\mathbb{P}^3$ containing $\mathscr{L}_0$. Donaldson [5] has shown that the restriction from $\mathbb{P}^3$ to $\mathbb{P}^2$ induces a one-to-one mapping from $M(r, c)$ onto the moduli space $\mathcal{O}M(r, c)$ of framed holomorphic bundles on $\mathbb{P}^2$. In the latter case no additional condition is required. We are going to make use of this result and concentrate on the description of $\mathcal{O}M(r, c)$.

Let us now elucidate the geometric interpretation of Takasaki's approach. S^4 is the manifold of real lines in $\mathbb{P}^3$. Let $\mathbb{G}_2$ be the Grassmann manifold whose points are projective lines in $\mathbb{P}^3$. $\mathbb{F}_{1,2} \subset \mathbb{P}^3 \times \mathbb{G}_2$ designates the flag ("coincidence") manifold. Fix a point $P_0 \in \mathscr{L}_0$. Then all the lines in $\mathbb{P}^3$ passing through P_0 form a submanifold $\mathbb{P}^2$ embedded into $\mathbb{G}_2$ and $\tilde{\mathbb{P}}^3 := \mathrm{pr}_2^{-1}(\mathbb{P}^2) \subset \mathbb{F}_{1,2}$ is the blow-up of $\mathbb{P}^3$ at the point P_0. Furthermore, relating to each point of $\mathbb{P}^3$ the unique real line passing through it one gets an embedding $\iota: \mathbb{P}^3 \hookrightarrow \mathbb{F}_{1,2}$. Summarizing, we get the following commutative diagram:



The projection π and the embeddings $\iota: \mathbb{P}^3 \hookrightarrow \mathbb{F}_{1,2}$ and $S^4 \hookrightarrow \mathbb{G}_2$ are real analytic; the other mappings in the diagram are holomorphic.

The usual step in the local approach is to use analytic continuation and replace four real variables by four independent complex variables. On the global level this means that S^4 is replaced by $\mathbb{G}_2$ and $\mathbb{P}^3$ by $\mathbb{F}_{1,2}$ and the analytical continuation is provided by the pull-back mapping pr_1^*. Actually, if a function f is holomorphic on $\mathbb{P}^3$, then it can be regarded as a real analytic function on $\iota(\mathbb{P}^3) \subset \mathbb{F}_{1,2}$ and the unique analytic continuation to $\mathbb{F}_{1,2}$ is $\mathrm{pr}_1^* f$.

The object which is then studied is a transition function G of the pulled back bundle on $\mathbb{F}_{1,2}$. Suppose that G is defined on a subdomain of $\mathrm{pr}_2^{-1}(\mathscr{U})$, where $\mathscr{U} \subset \mathbb{G}_2$ is an open set such that the fibration $\mathrm{pr}_2: \mathbb{F}_{1,2} \to \mathbb{G}_2$ is trivial over $\mathscr{U}$ with a coordinate λ on the fibre. In the general situation, the function $G(\lambda)$ is defined on a neighbourhood of the unit circle $\{|\lambda| = 1\}$ and one aims to find the Birkhoff decomposition [J is constant on the fibres, $W(0) = \hat{W}(\infty) = 1$]

$$G(\lambda)=\hat{W}(\lambda)^{-1}J^{-1}W(\lambda)\ .$$

On the global level, this can be achieved using the results about the loop group $L\,\mathrm{GL}(r,\mathbb{C})$, namely the existence of the diffeomorphism

$$L_1^-\mathrm{GL}(r,\mathbb{C})\times L^+\mathrm{GL}(r,\mathbb{C})\to L\,\mathrm{GL}(r,\mathbb{C})\ ,$$

with the image being open and dense in the component of the unit. Reference [6] serves as a nice source of information but we shall not pursue this topic further.

Takasaki's result can be rephrased in the following way. He proposed to consider the restriction $W^{(0)}:=W|\tilde{\mathbb{P}}^3$ and introduced the term "initial condition" for it. Let

$$\varphi:\mathbb{F}_{1,2}\to\tilde{\mathbb{P}}^3\ ,\ (P,\mathscr{L})\mapsto(P,\overline{PP_0})$$

be a rational mapping. It is singular on the submanifold $\mathrm{pr}_1^{-1}(P_0)\cong\mathbb{P}^2$. Then the pull-back $G=\varphi^*W^{(0)}$ together with the Birkhoff decomposition provides a solution to the initial value problem.

The central observation made in ref. [3] is that one can pick out from every equivalence class a unique solution such that

$$\hat{W}|\tilde{\mathbb{P}}^3\equiv\mathbf{1}\ ,\qquad J|\mathbb{P}^2\equiv\mathbf{1}$$

[here again $\mathbb{P}^2\cong\mathrm{pr}_1^{-1}(P_0)$]. The corresponding transition function G is called canonical and can be constructed in the following way. Let τ be the real structure on $\mathbb{P}^3$ and write $P_\infty:=\tau(P_0)$. Then the point P_∞ lies again on $\mathscr{L}_0$. Let us now recall Barth's result [7] which claims that the bundle is holomorphically trivial on almost all lines in $\mathbb{P}^3$. The exceptional lines are called jumping and if furnished with appropriate multiplicities, they form a divisor in $\mathbb{G}_2$ of degree c_2. As a corollary one can prove that the jumping lines passing through a fixed point, say P_0, form a divisor in $\mathbb{P}^2\subset\mathbb{G}_2$ of degree c_2 (corollary 4.2 in ref. [3]). Denote by $\mathscr{S}_0$ (resp. $\mathscr{S}_\infty$) the union of all jumping lines containing the point P_0 (resp. P_∞) and put $\mathscr{U}_0=\mathbb{P}^3\setminus\mathscr{S}_\infty$, $\mathscr{U}_\infty=\mathbb{P}^3\setminus\mathscr{S}_0$. (Then $P_0\in\mathscr{U}_0$, $P_\infty\in\mathscr{U}_\infty$, for the bundle is holomorphically trivial on $\mathscr{L}_0$.) One can continue the distinguished trivialization on $\mathscr{L}_0$ along all the lines that are not jumping and pass through P_∞ (resp. P_0) to get a holomorphic trivialization $\{s_j\}$ (resp. $\{\hat{s}_j\}$) on $\mathscr{U}_0$ (resp. $\mathscr{U}_\infty$). The corresponding transition function

$$G=(G_{jk})\ ,\qquad s_k=\sum_j\hat{s}_jG_{jk}$$

is the desired "canonical transition function" (CTF).

As we have already mentioned we are going to make use of Donaldson's restriction. The same construction can be reproduced verbatim for framed holomorphic vector bundles on $\mathbb{P}^2$. The set $\mathscr{S}_0$ (resp. $\mathscr{S}_\infty$) is then a union of finitely many jumping lines and provided we furnish them with multiplicities according to

Barth's result, the number of these lines exactly equals c_2. Choose homogeneous coordinates (z_0, z_1, z_2) on $\mathbb{P}^2$ in such a way that the line $\mathscr{L}_0$ is determined by the equation $z_0=0$ and the points P_0, P_∞ are determined by the equations $z_0=z_1=0$ and $z_0=z_2=0$, respectively. The CTF is known to be unimodular and to have the following form:

$$G(z)=1+\frac{1}{\not{p}_0(z)\not{p}_\infty(z)}\,\mathscr{R}(z)\,,$$

$$\not{p}_0(z)=z_1^c+\sum_{j=1}^{c} s_j z_0^j z_1^{c-j}\,,$$

$$\not{p}_\infty(z)=z_2^c+\sum_{k=1}^{c} t_k z_0^k z_2^{c-k}\,,$$

$$\mathscr{R}(z)=\sum_{j,k=1}^{c} R_{jk} z_0^{j+k} z_1^{c-j} z_2^{c-k}\,, \tag{1.1}$$

where s_j, $t_k \in \mathbb{C}$, $R_{jk} \in \mathbb{C}^{r,r}$. The polynomials $\not{p}_0(z)$ and $\not{p}_\infty(z)$ are unambiguously specified by the condition that the zero sets $\not{p}_0(z)=0$ and $\not{p}_\infty(z)=0$ coincide with $\mathscr{S}_0$ and $\mathscr{S}_\infty$ (including the multiplicities), respectively.

The construction is gauge independent and the mapping

$$\mathscr{O}M(r, c) \ni [F] \mapsto (s_j, t_k R_{jk}) \in \mathbb{C}^N, N=2c+r^2c^2\,,$$

is holomorphic and injective and the image is a locally algebraic set in $\mathbb{C}^N$. This means that there exists an irreducible [recall that $\mathscr{O}M(r, c)$ is connected] algebraic set $\mathscr{A}(r, c)$ in $\mathbb{C}^N$ containing $\mathscr{O}M(r, c)$ and both sets have the same complex dimension, $\dim \mathscr{O}M(r, c)=\dim \mathscr{A}(r, c)=2rc$ (for c sufficiently large). To get $\mathscr{O}M(r, c)$ one has to remove from $\mathscr{A}(r, c)$ some closed subset of singular points but this paper is not concerned with these singularities.

After recalling the final result stated in ref. [3] we are able to formulate properly what is the goal of the present paper. The task is to describe explicitly the algebraic sets $\mathscr{A}(2, c)$, $c=2, 3, \ldots$ (the one-instanton case is transparent and well known), i.e., we aim to derive the corresponding algebraic equations. This task splits into two steps. First, one is concerned with the problem of removability of singularities. Let G be a matrix-valued function on $\mathbb{P}^2$ having the form (1.1). If regarded as a transition function, G determines a vector bundle $\tilde{F}_G$ on

$$\mathscr{W}=(\mathbb{P}^2\setminus\{\not{p}_0(z)=0\})\cup(\mathbb{P}^2\setminus\{\not{p}_\infty(z)=0\})$$
$$=\mathbb{P}^2\setminus\{\not{p}_0(z)=\not{p}_\infty(z)=0\}\,.$$

Let $\imath: \mathscr{W} \hookrightarrow \mathbb{P}^2$ be the embedding and $\tilde{\mathscr{F}}_G$ be the sheaf of germs of holomorphic sections in $\tilde{F}_G$. It can be shown rather easily that the direct image $\mathscr{F}_G := \imath_* \tilde{\mathscr{F}}_G$ is a coherent sheaf on $\mathbb{P}^2$. The question to be answered is whether $\mathscr{F}_G$ is a locally free sheaf corresponding to some vector bundle F_G on $\mathbb{P}^2$. The problem is local and can be reduced to an assertion about separation of singularities of a meromorphic matrix-valued function defined on $\mathbb{C}^2$. It turns out that for arbitrary rank r the singularities are always removable. Second, from the construction it follows that the CTF corresponding to $\mathscr{F}_G$ coincides with G as a function on $\mathbb{P}^2$. But this does not mean that G is already written in the canonical form. It is true that the zero set of the polynomial $\not{p}_0(z)$ coincides with the set of jumping lines $\mathscr{S}_0$ but the multiplicities may differ. Hence the second Chern class $c_2(F_G)$ must be computed. The equality

$$c_2(F_G)=c$$

then provides the desired algebraic equations.

The result suggests a parameterization of the moduli space $\mathcal{O}M(2,c)$. To be specific, the two-instanton case is considered more closely in the appendix. This case is distinguished by the fact that it is not as trivial as the one-instanton case is but the number of involved parameters is low enough so that one is able to handle them. We found by error and trial the explicit form of the inversion of Donaldson's restriction mapping (which is not known in the general case) and in this way obtained a convenient parameterization of the two-instanton case, distinct from those being used before (such as the modified 't Hooft solutions).

2. Removability of singularities

Let G have the form (1.1). Consider a point $Q\in\mathbb{P}^2$ such that

$$\not{p}_0(z_Q)=\not{p}_\infty(z_Q)=0\,.$$

We note that two jumping lines in $\mathbb{P}^2$ such that one of them contains the point P_0 and the other contains the point P_∞ intersect transversally. Otherwise both projective lines would coincide and this is a contradiction for the line $\overline{P_0P_\infty}$ is not jumping. Consequently, the sets $\not{p}_0(z)=0$ and $\not{p}_\infty(z)=0$ are smooth and transverse in some neighbourhood of the singular point Q and so Q is an isolated intersection point. Owing to a consequence of Hartog's theorem (which guarantees the removability of singularities of a complex analytic function provided the singular points are contained in an analytic set of codimension at least two) the discussion can be simplified. The singularity in the point Q is removable if and only if the bundle $\tilde{F}_G$ is holomorphically trivial on some punctured neighborhood $\mathscr{U}\setminus\{Q\}$ of the point Q. The situation can be studied locally. According to the above remark one can choose coordinates ξ, η on a neighbourhood $\mathscr{U}$ of the point Q in such a way that the set $\not{p}_0(z)=0$ (resp. $\not{p}_\infty(z)=0$) coincides on $\mathscr{U}$ with $\xi=0$ (resp. $\eta=0$) and hence Q has the coordinates $\xi=\eta=0$. The vector bundle $\tilde{F}_G$ is holomorphically trivial on $\mathscr{U}\setminus\{Q\}$ if and only if the poles of the transition function G can be separated, as the following theorem asserts.

Theorem 2.1. *Let $G(\xi,\eta)$ be an $r\times r$ matrix-valued function defined on a neighbourhood of the origin in $\mathbb{C}^2$. Suppose that G is unimodular and meromorphic with the poles on the set $\xi\eta=0$. Then there exists a decomposition $G=XY$, where X and Y are again unimodular and meromorphic matrix-valued functions and X (resp. Y) has the pole on the line $\xi=0$ (resp. $\eta=0$).*

We can suppose that the order K of the pole of $G(\xi,\eta)$ on the line $\eta=0$ is positive (otherwise there is nothing to be proved) and proceed by induction. So it is sufficient to prove the induction step.

Induction step. There exists a matrix-valued function $H(\xi,\eta)$ which is unimodular and meromorphic with the pole on the line $\eta=0$ and such that the product GH has the poles again on the set $\xi\eta=0$ but the order of the pole on the line $\eta=0$ is equal at most to $(K-1)$.

The proof is based on the following series of lemmas.

Lemma 2.2. *Let $\mathfrak{k}$ be a field and $Z_j\in\mathfrak{k}^{r,r}$, $j=0,1,\ldots,\nu-1$, be some matrices. Put*

$$Z(\lambda)=\sum_{j=0}^{\nu-1} Z_j\lambda^j,$$

$$\mathfrak{Z}=\begin{pmatrix} Z_0 & 0 & \ldots & 0\\ Z_1 & Z_0 & \ldots & 0\\ \vdots & \vdots & & \vdots\\ Z_{2\nu-1} & Z_{2\nu-2} & \ldots & Z_0\end{pmatrix}.$$

Let σ be the order of the zero of $\det Z(\lambda)$ *at the point $\lambda=0$. Then*

$$\sigma\geqslant\nu\quad\text{iff}\dim\ker(\mathfrak{Z})\geqslant\nu\,.$$

Proof. Put

$$\mathfrak{V}_0=\mathfrak{k}^r\oplus\cdots\oplus\mathfrak{k}^r\quad(\nu\text{ copies})\,,$$

and let $\pi_j:\mathfrak{V}_0\to\mathfrak{k}^r$, $j=0,1,\ldots,\nu-1$, be the projection onto the jth direct summand Further, put

$$\mathfrak{V}_j=\{\varphi\in\mathfrak{V}_0;\ \pi_0(\varphi)=\cdots=\pi_{j-1}(\varphi)=0\}\,,\quad 1\leqslant j\leqslant\nu\,.$$

So we have filtrations

$$\mathfrak{V}_0\supset\mathfrak{V}_1\supset\cdots\supset\mathfrak{V}_\nu=0\,,$$

$$\ker(\mathfrak{Z})=\mathfrak{K}\supset\mathfrak{K}_1\supset\cdots\supset\mathfrak{K}_\nu=0\,,\quad \mathfrak{K}_j:=\ker(\mathfrak{Z})\cap\mathfrak{V}_j\,.$$

Let E be the shift operator on $\mathfrak{B}_0$,

$$E(f_0, f_1, \dots, f_{\nu-1}) = (0, f_0, \dots, f_{\nu-2}) .$$

Clearly, $E(\mathfrak{R}_j) \subset \mathfrak{R}_{j+1}$ and the induced mapping (denoted by the same letter) $E: \mathfrak{R}_j/\mathfrak{R}_{j+1} \to \mathfrak{R}_{j+1}/\mathfrak{R}_{j+2}$ is injective. Moreover, using the projection π_j, one can embed the factor space $\mathfrak{R}_j/\mathfrak{R}_{j+1}$ into $\mathfrak{f}^r$ and after this embedding the mapping E becomes identical and we thus have the inclusions

$$\mathfrak{R}_0/\mathfrak{R}_1 \subset \mathfrak{R}_1/\mathfrak{R}_2 \subset \cdots \subset \mathfrak{R}_{\nu-1}/\mathfrak{R}_\nu \subset \mathfrak{f}^r .$$

Choose direct summands $\mathfrak{L}_j$: $\mathfrak{R}_{j+1}/\mathfrak{R}_{j+2} = \mathfrak{R}_j/\mathfrak{R}_{j+1} \oplus \mathfrak{L}_{j+1}$, for $0 \leqslant j \leqslant \nu-2$, $\mathfrak{L}_0 = \mathfrak{R}_0/\mathfrak{R}_1$ by definition, and put $l_j = \dim \mathfrak{L}_j$. Then

$$\mathrm{Ker}(Z_0) = \mathfrak{R}_{\nu-1}/\mathfrak{R}_\nu = \mathfrak{L}_0 \oplus \mathfrak{L}_1 \oplus \cdots \oplus \mathfrak{L}_{\nu-1} ,$$

$$\dim \mathrm{Ker}(Z_0) = l_0 + l_1 + \cdots + l_{\nu-1} ,$$

$$\begin{aligned} \dim \mathrm{Ker}(\mathfrak{Z}) &= \dim(\mathfrak{R}_0/\mathfrak{R}_1) + \dim(\mathfrak{R}_1/\mathfrak{R}_2) + \cdots + \dim(\mathfrak{R}_{\nu-1}/\mathfrak{R}_\nu) \\ &= \nu l_0 + (\nu-1) l_1 + \cdots + l_{\nu-1} . \end{aligned}$$

Choose a basis $\{f_{jk};\ 1 \leqslant k \leqslant l_j\}$ in the space $\mathfrak{L}_j$ and the vectors $\varphi_{jk} \in \mathfrak{R}_j$ such that $\pi_j(\varphi_{jk}) = f_{jk}$. Further, choose a direct summand $\mathfrak{S}$: $\mathfrak{f}^r = \mathrm{Ker}(Z_0) \oplus \mathfrak{S}$, and a basis $\{g_1, \dots, g_d\}$ in $\mathfrak{S}$. Set

$$F_{jk}(\lambda) = \lambda^{-\nu} \sum_{m=0}^{\nu-1} \lambda^m \pi_m(\varphi_{jk}) ,$$

$$F(\lambda) = [(F_{jk}(\lambda))_{\substack{0 \leqslant j \leqslant \nu-1 \\ 1 \leqslant k \leqslant l_j}}, (Z_0 g_j)_{1 \leqslant j \leqslant d}] ,$$

$$\begin{aligned} h_{jk} &= Z(\lambda) F_{jk}(\lambda)|_{\lambda=0} \\ &= Z_{\nu-j} f_{jk} + Z_{\nu-j-1} \pi_{j+1}(\varphi_{jk}) + \cdots + Z_1 \pi_{\nu-1}(\varphi_{jk}) , \end{aligned}$$

for $1 \leqslant j \leqslant \nu-1$, $1 \leqslant k \leqslant l_j$. By construction, the vectors $\{(h_{jk})_{1 \leqslant j \leqslant \nu-1, 1 \leqslant k \leqslant l_j}, (Z_0 g_j)_{1 \leqslant j \leqslant d}\}$ are independent in $\mathfrak{f}^r$.

Summarizing, one can draw the following conclusions.

(1) $\det(Z(\lambda)F(\lambda))$ does not contain negative powers of λ. On the other hand, the lowest power of λ appearing with a nonzero coefficient in $\det F(\lambda)$ is

$$-(\nu l_0 + (\nu-1) l_1 + \cdots + l_{\nu-1}) = -\dim \ker(\mathfrak{Z}) .$$

Hence $\sigma \geqslant \dim \ker(\mathfrak{Z})$.

(2) If $l_0 = \dim(\mathfrak{R}_0/\mathfrak{R}_1) = 0$, then $\sigma = \dim \ker(\mathfrak{Z})$. Actually, $l_0 = 0$ implies that

$$\det(Z(\lambda)F(\lambda)) = \det((h_{jk})_{j,k}, (Z_0 g_j)_j) + O(\lambda) ,$$

and hence $\sigma - \dim \ker(\mathfrak{Z}) = 0$.

(3) If $\dim \ker(\mathfrak{Z}) < \nu$, then $l_0 = 0$ and, consequently, $\sigma = \dim \ker(\mathfrak{Z})$. Actually, $\nu l_0 \leqslant \dim \ker(\mathfrak{Z}) < \nu$ is possible only if $l_0 = 0$.

This proves the lemma. □

Let $\mathfrak{R}$ be a local Noetherian ring with a maximal ideal $\mathfrak{m}$. Denote by $\mathfrak{k}=\mathfrak{R}/\mathfrak{m}$, $\mathfrak{f}=\mathfrak{m}^{-1}\mathfrak{R}$ the corresponding fields. We suppose that $\mathfrak{R}$ is a $\mathfrak{k}$-algebra, i.e., $\mathfrak{R}=\mathfrak{k}\oplus\mathfrak{m}$ and hence $\mathfrak{k}\subset\mathfrak{R}\subset\mathfrak{f}$. The symbol $\langle x_1, ..., x_k\rangle_A$ stands for the linear hull of elements $x_1, ..., x_k$ with coefficients from an Abelian group A. Let $\mathfrak{M}_0=\mathfrak{M}_1=\cdots=\mathfrak{M}_{\nu-1}\cong\mathfrak{R}^r$ be identical free $\mathfrak{R}$-modules and $\mathfrak{M}=\mathfrak{M}_0\oplus\mathfrak{M}_1\oplus\cdots\oplus\mathfrak{M}_{\nu-1}$ be their direct sum. $E:\mathfrak{M}\to\mathfrak{M}$ is again the shift operator. Let $\omega:\mathfrak{R}\to\mathfrak{k}$ (or $\omega:\mathfrak{R}^r\to\mathfrak{k}^r$) be the projection. For $\varphi=(f_0, ..., f_{\nu-1})\in\mathfrak{M}$ such that $\omega(\varphi)\neq 0$, the number

$$\mu(\varphi):=\nu-1-\min\{k;\ \omega(f_k)\neq 0\}$$

will be called the weight of the element φ and we put

$$\gamma(\varphi):=\omega(f_{\nu-1-\mu})\in\mathfrak{k}^r,\quad \mu=\mu(\varphi).$$

Lemma 2.3. *Suppose there is given a collection $\Phi=\{\varphi_1, ..., \varphi_\sigma\}$ of elements from the module $\mathfrak{M}$ satisfying*

(1) $E\varphi_k\in\{\varphi_1, \varphi_2, ..., \varphi_{k-1}\}$, for $k=1, ..., \sigma$,

(2) Φ is independent over $\mathfrak{f}$.

Then there exists a collection $\Psi=\{\psi_1, ..., \psi_\sigma\}$ of elements from $\mathfrak{M}$ satisfying

(I) there exist integers $s_j\in\mathbb{N}_0$ such that

$$\varphi_j\in\mathfrak{m}^{s_j}\psi_j+\langle\psi_1, ..., \psi_{j-1}\rangle_{\mathfrak{R}},\quad \textit{for all } j;$$

(II) all the vectors $\omega(\psi_j)$ are nonzero and for each $s\in\{0, ..., \nu-1\}$ the collection

$$\Gamma(s;\Psi):=\{\gamma(\psi_j);\ \psi_j\in\Psi \textit{ and } \mu(\psi_j)=s\}$$

is independent over $\mathfrak{k}$ (or empty);

(III) $E\psi_j\in\langle\psi_1, ..., \psi_{j-1}\rangle_{\mathfrak{f}}$, for all j.

Remark. From (I) it follows that

$$\text{(I}'\text{)}\qquad \begin{aligned}&\psi_j\in\mathfrak{m}^{-s_j}\varphi_j+\langle\varphi_1, ..., \varphi_{j-1}\rangle_{\mathfrak{f}},\\ &\langle\psi_1, ..., \psi_j\rangle_{\mathfrak{f}}=\langle\varphi_1, ..., \varphi_j\rangle_{\mathfrak{f}},\quad \text{for all } j.\end{aligned}$$

Proof. Observe that (III) is a consequence of (I′) and property (1) of Φ. Thus it is enough to verify only conditions (I) and (II). We shall construct Ψ step by step requiring the collections $\Psi_j:=\{\psi_1, ..., \psi_j\}$ to satisfy (I) and (II). For $j=0$, Ψ_0 is empty. Let $j>0$ and suppose Ψ_{j-1} has been found. By Krull's Intersection Theorem, there exists a maximal integer $s_j\in\mathbb{N}_0$ such that

$$\varphi_j\in\mathfrak{m}^{s_j}\mathfrak{M}+\langle\Psi_{j-1}\rangle_{\mathfrak{R}}.$$

This means that there exists $\psi_j\in\mathfrak{M}$ such that condition (I) holds (for this j). By the choice of s_j, $\omega(\psi_j)$ is nonzero. One can select ψ_j in such a way that the weight $\mu=\mu(\psi_j)$ is the lowest possible. It remains to verify that Ψ_j satisfies (II). If $s\neq\mu$, then $\Gamma(s;\Psi_j)=\Gamma(s;\Psi_{j-1})$ and condition (II) is valid. For $s=\mu$, $\Gamma(\mu;\Psi_j)=\{\Gamma(\mu;\Psi_{j-1}),\gamma(\psi_j)\}$ and (II) is again valid due to the minimality condition imposed on $\mu(\psi_j)$. □

Suppose $\Psi=\{\psi_1,\dots,\psi_\sigma\}$ satisfies (I), (II), (III). For an element $\psi_j=(f_{j,0},\dots,f_{j,\nu-1})$ of Ψ we put

$$Q_j(\lambda)=\sum_{k=0}^{\nu-1} f_{jk}\lambda^{\nu-1-k}\in\mathfrak{R}^r[\lambda]\,. \tag{2.1}$$

The factor mapping ω can be extended as a ring homomorphism, $\omega:\mathfrak{R}^r[\lambda]\to\mathfrak{k}^r[\lambda]$. For an arbitrary m-tuple of indices $j_1,\dots,j_m$, the external products

$$Q_{j_1}\wedge\cdots\wedge Q_{j_m},\qquad \omega(Q_{j_1})\wedge\cdots\wedge\omega(Q_{j_m}) \tag{2.2}$$

can be regarded as polynomials with coefficients from the space $\wedge\,\mathfrak{R}^r$ and $\wedge\,\mathfrak{k}^r$, respectively.

Lemma 2.4.

(i) For an arbitrary m-tuple of mutually different indices $j_1, j_2,\dots,j_m$, the degree of the polynomial $Q_{j_1}\wedge\cdots\wedge Q_{j_m}$ is less than or equal to $\max\{j_1,\dots,j_m\}-m$.

(ii) Suppose that a subcollection $\Psi'=\{\psi_{j_1},\dots,\psi_{j_m}\}$, $j_1<\cdots<j_m$, satisfies:

(a) $\{\Gamma(0;\Psi'),\Gamma(1;\Psi'),\dots,\Gamma(\nu-1;\Psi')\}$ is independent over $\mathfrak{k}$,

(b) $\langle\Gamma(s;\Psi');k\leqslant s\leqslant\nu-1\rangle_{\mathfrak{k}}=\langle\Gamma(s;\Psi);k\leqslant s\leqslant\nu-1\rangle_{\mathfrak{k}}$, for all $k=0,1,\dots,\nu-1$. Then the degrees of the polynomials (2.2) are the same and equal to $\sigma-m$.

Proof.

(i) We can suppose that the indices are ordered in increasing order. The shift operator E can be naturally redefined, $E:\mathfrak{R}^r[\lambda]\to\mathfrak{R}^r[\lambda]$,

$$E\left(\sum_{j=0}^{\nu-1} f_j\lambda^{\nu-1-j}\right)=\sum_{j=1}^{\nu-1} f_{j-1}\lambda^{\nu-1-j}\,.$$

Property (III) of Ψ means that

$$EQ_j\in\langle Q_1,\dots,Q_{j-1}\rangle_{\mathfrak{k}}\,,\quad \text{for } j=1,\dots,\sigma\,. \tag{2.3}$$

Owing to (2.3) and since $Q_j(\lambda)=Q_j(0)+\lambda EQ_j(\lambda)$, one finds that

$$Q_1(\lambda)\wedge\cdots\wedge Q_k(\lambda)=Q_1(0)\wedge\cdots\wedge Q_k(0)\,, \tag{2.4}$$

for $k=0,1,\dots,\sigma$. Now, put $S=j_m-m$. To prove (i) we proceed by induction on S. If $S=0$ then $j_1=1,\dots,j_m=m$, and (2.4) proves the assertion. If $S>0$, then $j_m>m$ and it suffices to show that the degree of the polynomial

$$Q_{j_1}(\lambda)\wedge\cdots\wedge Q_{j_{m-1}}(\lambda)\wedge EQ_{j_m}(\lambda)$$

is less than S. But with respect to (2.3), this is already a consequence of the induction hypothesis.

(ii) Denote $\epsilon_j=|\Gamma(j;\Psi)|$, $\epsilon'_j=|\Gamma(j;\Psi')|$. Clearly, $\sum\epsilon_j=\sigma$ and conditions (a), (b) imply that

$$\epsilon'_k+\epsilon'_{k+1}+\cdots+\epsilon'_{\nu-1}\geqslant\epsilon_k\,,\quad \text{for } k=0,1,\ldots,\nu-1\,.$$

Summing these inequalities one gets

$$\sum_{j=0}^{\nu-1} j\epsilon'_j\geqslant\sigma-m\,. \tag{2.5}$$

But by condition (a), the l.h.s. of (2.5) is equal to the degree of the polynomial $\omega(Q_{j_1})\wedge\cdots\wedge\omega(Q_{j_m})$. Using part (i) of the present lemma one finds that

$$\deg(Q_{j_1}\wedge\cdots\wedge Q_{j_m})\leqslant\sigma-m\leqslant\deg(\omega(Q_{j_1})\wedge\cdots\wedge\omega(Q_{j_m}))\,.$$

But the homomorphism ω cannot increase the degree of the polynomial and hence the equality signs must hold. □

Proof of the induction step. Let us write $G(\xi,\eta)$ in the form

$$G(\xi,\eta)=\eta^{-K}\sum_{j=0}^{\infty}G_j(\xi)\eta^j\,,$$

and construct a blockwise matrix

$$\mathfrak{Z}=\begin{pmatrix} G_0 & 0 & \cdots & 0\\ G_1 & G_0 & \cdots & 0\\ \vdots & \vdots & & \vdots\\ G_{r-1} & G_{r-2} & \cdots & G_0\end{pmatrix}.$$

The unimodularity of G and the fact that $K>0$ imply that the order of zero at the point $\eta=0$ of the function $\det(G_0+G_1\eta+\cdots+G_{r-1}\eta^{r-1})$ is not less than r. Let $\mathfrak{R}=\mathcal{O}_0=\mathbb{C}\{\xi\}$ be the local ring of germs of holomorphic functions at the point $\xi=0$. Then $\mathfrak{k}=\mathbb{C}$ and $\mathfrak{f}$ is the field of germs of meromorphic functions. Lemma 2.2 implies that there exists a collection $\Phi=\{\varphi_1,\ldots,\varphi_r\}$ of elements of $\mathfrak{R}^r\oplus\cdots\oplus\mathfrak{R}^r$ such that $\mathfrak{Z}\varphi_j=0$ for all j and, moreover, properties (1), (2) in lemma 2.3 (with $\sigma=r$) are satisfied. Lemma 2.3 ensures the existence of a collection $\Psi=\{\psi_1,\ldots,\psi_r\}$ satisfying properties (I), (II), (III). Now, deleting subsequently according to decreasing weight elements from Ψ one arrives at a collection $\Psi'=\{\psi_{j_1},\ldots,\psi_{j_m}\}$ satisfying the properties (a), (b) from lemma 2.4 ad (ii). The polynomials $Q_j(\eta^{-1})\in\mathfrak{R}^r[\eta^{-1}]$ defined by (2.1) can be regarded as germs of meromorphic vector-valued functions at the point $\xi=\eta=0$. Their poles lie on the line $\eta=0$. Let us denote these meromorphic functions by

$$H_k(\xi, \eta) = Q_{jk}(\xi, \eta^{-1}), \quad \text{for } k = 1, \ldots, m.$$

Since $\mathfrak{Z}\psi_j = 0$ for all j, the order of the pole on the line $\eta = 0$ of the meromorphic function $G(\xi, \eta)H_k(\xi, \eta)$ is equal at most to $(K-1)$. One can complete the collection $\{\gamma(\psi_{j_1}), \ldots, \gamma(\psi_{j_m})\}$ by vectors $\{h_{m+1}, \ldots, h_r\}$ to a basis in $\mathbb{C}^r$ and put

$$H_k(\xi, \eta) = h_k \eta, \quad \text{for } k = m+1, \ldots, r.$$

Let $H'(\xi, \eta)$ be the matrix with columns $H_k(\xi, \eta)$. From lemma 2.4 ad (ii) it follows that $\kappa(\xi, \eta) := \det H'(\xi, \eta)$ does not contain negative powers of η and hence is holomorphic in a neighbourhood of the origin in $\mathbb{C}^2$. Clearly, $\kappa(0, 0) \neq 0$. Then the matrix-valued function

$$H(\xi, \eta) = H'(\xi, \eta) \operatorname{diag}(\kappa(\xi, \eta)^{-1}, 1, \ldots, 1)$$

will do and can be used in the induction step. □

3. Computation of the topological charge

In what follows, we restrict ourselves to rank-two bundles, i.e., the gauge group is assumed to be SU(2). For a 2×2 matrix A we shall use the notation

$$A^0 = \operatorname{tr}(A)\, \mathbf{1} - A.$$

Hence $(AB)^0 = B^0 A^0$ and $A^0 A = A A^0 = \det(A)\, \mathbf{1}$. The matrix-valued function G is again assumed to have the form (1.1) and in virtue of theorem 2.1 it determines a holomorphic rank-2 bundle F_G on $\mathbb{P}^2$.

Barth's result about jumping lines suggests a way to compute the second Chern class $c_2(F_G)$. Let $\tilde{\mathbb{P}}^2 \subset \mathbb{P}^2 \times \mathbb{P}^1$ be the blow-up of $\mathbb{P}^2$ at the point P_0. $\mathbb{P}^1$ is assumed to be embedded into $\mathbb{P}^2$ as the line determined by the equation $z_2 = 0$. Then the sheaf $\mathscr{E} := R^1 \mathrm{pr}_{2*} \mathrm{pr}_1^* \mathscr{F}_G(-1)$ [the first direct image of the pulled back twisted sheaf $\mathrm{pr}_1^* \mathscr{F}_G(-1)$] has a discrete support and the sum of the dimensions of the stalks is equal to $c_2(F_G)$. This means that this sheaf determines a divisor $\mathscr{D}$ in $\mathbb{P}^1$ such that $\operatorname{supp} \mathscr{D} = \operatorname{supp} \mathscr{E}$ and $\deg \mathscr{D} = c_2$. For $Q \in \operatorname{supp} \mathscr{E} \subset \mathbb{P}^1$, denote by $\sigma(Q)$ the dimension of the stalk over the point Q. By definition, $\sigma(Q)$ is the multiplicity of the jumping line $\overline{P_0 Q}$. Thus,

$$c_2 = \sum \sigma(Q),$$

where the sum runs over all jumping lines $\overline{P_0 Q}$.

One can use $\xi = z_1/z_0$ as a coordinate on the projective line $\mathbb{P}^1 \subset \mathbb{P}^2$. The value $\xi = \infty$ corresponds to the point $P_\infty \in \mathbb{P}^1$. Our next task is to compute the multiplicity $\sigma(x)$ of a jumping line $\mathscr{L} = \overline{P_0 Q}$ where the value $\xi = x$ corresponds to the point $Q \in \mathbb{P}^1$. Then one can use $\eta = z_2/z_0$ as a coordinate on the line $\mathscr{L}$. The value $\eta = \infty$ corresponds to the point P_0. After the blow-up at the point P_0, one chooses a pencil neighbourhood of the jumping line and faces the following situation. Let $\mathscr{U}$ be a sufficiently small neighbourhood of the point Q in $\mathbb{P}^1$ and $\mathscr{V}_1$, $\mathscr{V}_2$ be two open subsets of $\mathscr{U}\times\mathscr{L}$ determined by the inequalities $\eta\neq 0$, $\eta\neq y_1, ..., \eta\neq y_c$ and $\xi\neq x$, $\eta\neq\infty$, respectively, where $\{y_j\}$ are the roots of the polynomial $p_\infty(\eta)$. The matrix-valued function $\eta^k G(\xi,\eta)$, $k\in\mathbb{Z}$, is well defined on $\mathscr{V}_1\cap\mathscr{V}_2$ and as a transition function gluing the trivial bundle on $\mathscr{V}_1$ to that on $\mathscr{V}_2$, determines a twisted vector bundle $F(k)$ on $\mathscr{V}_1\cup\mathscr{V}_2$. The singular points corresponding to the values $\xi=0$ and $\eta=0, y_1, ..., y_c$, are removable. The multiplicity $\sigma(x)$ then equals the dimension of the first cohomology group,

$$\sigma(x)=\dim H^1(\mathscr{U}\times\mathscr{L},\mathscr{F}(-1))\,.$$

A convenient way to compute $\sigma(x)$ is to use a resolution for $\mathscr{F}(-1)$. After translation $\xi=\xi'+x$, one can suppose without loss of generality that $x=0$. Let us express the matrix-valued function $G(\xi,\eta)$ in the form

$$G(\xi,\eta)=\mathbf{1}+\frac{1}{\xi^\nu p(\xi)q(\eta)}\,\mathscr{P}(\xi,\eta)\,,$$

where $\nu>0$ [the case $\nu=0$ is trivial: $\sigma(0)=0$], $p(0)\neq 0$,

$$q(\eta)=\eta^\mu+\tau_1\eta^{\mu-1}+\cdots+\tau_\mu=(\eta-y_1)^{\mu_1}\cdots(\eta-y_n)^{\mu_n}\,,$$

$\mu=\mu_1+\cdots+\mu_n$, the roots $y_1, ..., y_n$ are mutually different and $\mathscr{P}(0,\eta)\neq 0$, $\mathscr{P}(\xi,y_j)\neq 0$, $1\leqslant j\leqslant n$. As $G(P_0)=G(P_\infty)=\mathbf{1}$, the degree of the matrix-valued polynomial $\mathscr{P}(\xi,\eta)$ in the variable ξ with η being fixed (resp. in the variable η with ξ being fixed) is less than $\nu+\deg p$ (resp. μ). The columns of the 2×4 matrix-valued function

$$r_1(\xi,\eta)=(\xi^\nu\eta^{-\mu}G^{-1}(\xi,\eta),\eta^{-\mu}q(\eta)\,\mathbf{1})$$

represent a quadruple of holomorphic sections of the vector bundle $F(\mu)$ in the trivialization on the open set $\mathscr{V}_1$. The same holomorphic sections in the trivialization on $\mathscr{V}_2$ are given by the columns of the matrix-valued function

$$r_2(\xi,\eta)=\eta^\mu G(\xi,\eta)r_1(\xi,\eta)=(\xi^\nu\,\mathbf{1},q(\eta)G(\xi,\eta))\,.$$

The isolated singular points are again removable, and the quadruple $r(\xi,\eta)$ of holomorphic sections generates $F(\mu)$ [i.e., rank $r(\xi,\eta)=2$] *everywhere on $\mathscr{U}\times\mathscr{L}$, possibly except the critical points* $(\xi,\eta)=(0,y_1), ..., (0,y_n)$.

Lemma 3.1. *The quadruple of holomorphic sections $r(\xi,\eta)$ generates $F(\mu)$ in the critical point $(\xi,\eta)=(0,y_j)$ if and only if*

$$\xi^\nu(\eta-y_j)^{\mu_j}G(\xi,\eta)\,|_{\xi=0,\eta=y_j}\neq 0\,. \tag{3.1}$$

Proof. We can suppose without loss of generality that $y_j=0$ and omit in this proof the subscript j in μ_j. We know that there exists a decomposition $G=XY$. The order of the pole of the matrix-valued function X on the line $\xi=0$ is equal to ν and the order of the pole of Y on the line $\eta=0$ is equal to μ. Then the columns of the function X represent a frame of $F(\mu)$ in some neighbourhood of the point $(\xi,\eta)=(0,0)$ in the trivialization on $\mathscr{V}_2$. In this frame the quadruple of holomorphic sections is expressed as

$$X^{-1}(\xi,\eta)\,r_2(\xi,\eta)=(\xi^{\nu}X^{-1}(\xi,\eta),\eta^{\mu}Y(\xi,\eta))\,.$$

Since $\operatorname{rank}(\xi^{\nu}X^{-1})_{\xi=0}\geqslant 1$, $\operatorname{rank}(\eta^{\mu}Y)_{\eta=0}\geqslant 1$, we have

$$\operatorname{rank}(\xi^{\nu}X^{-1}(\xi,\eta),\eta^{\mu}Y(\xi,\eta))_{\xi=0,\eta=0}=2$$

if and only if

$$(\xi^{\nu}X^{-1})^{0}(\eta^{\mu}Y)\,|_{\xi=0,\eta=0}\neq 0\,.$$

Now it suffices to notice that the unimodularity of X implies $(X^{-1})^{0}=X$. □

Throughout the rest of this section, condition (3.1) is assumed to be satisfied. So we get a resolution

$$0\to\mathscr{K}\to\mathscr{O}(\mathscr{U}\times\mathscr{L})^4\to\mathscr{F}(\mu)\to 0\,,$$

and taking the first direct image we obtain

$$\begin{aligned}0\to R^1\pi_*\mathscr{K}(-\mu-1)\overset{\kappa}{\to}R^1\pi_*\mathscr{O}(-\mu-1)^4\\ \to R^1\pi_*\mathscr{F}(-1)\to 0\,,\end{aligned}$$

where $\pi\colon\mathscr{U}\times\mathscr{L}\to\mathscr{U}$ is a projection. It is known (cf. ref. [7]) that $R^1\pi_*\mathscr{K}(-\mu-1)$ and $R^1\pi_*\mathscr{O}(-\mu-1)^4$ are locally free sheaves on $\mathscr{U}$ having the same dimensions equal to 4μ, and if the mapping

$$\kappa\colon H^1(\mathscr{U}\times\mathscr{L},\mathscr{K}(-\mu-1))\to H^1(\mathscr{U}\times\mathscr{L},\mathscr{O}(-\mu-1)^4)$$

is expressed in some basis then the order of zero in the point $\xi=0$ of the function $\det\kappa(\xi)$ is equal to $\sigma(0)$.

The complex manifold $\mathscr{U}\times\mathscr{L}$ can be covered by two Stein sets $\mathscr{W}_1$, $\mathscr{W}_2$ determined by the conditions $|\eta|>M$ (M large enough) and $\eta\neq\infty$, respectively. Consequently, Leray's theorem on cohomology is applicable. For any coherent sheaf $\mathscr{S}$ on $\mathscr{U}\times\mathscr{L}$, the vector space $H^1(\mathscr{U}\times\mathscr{L},\mathscr{S})$ consists of holomorphic sections defined on $\mathscr{W}_1\cap\mathscr{W}_2$ modulo sections having the form $\psi_1-\psi_2$, where ψ_j is the restriction of a holomorphic section on $\mathscr{W}_j$, $j=1,2$. Writing

$$\mathscr{O}(-\mu-1)^4=\mathscr{O}(-\mu-1)^2\oplus\mathscr{O}(-\mu-1)^2\,,$$

we choose a basis in $H^1(\mathscr{U}\times\mathscr{L},\mathscr{O}(-\mu-1)^4)$ (expressed in the trivialization on

$\mathscr{W}_2$) as follows:

$$\{(\eta^{-j}e_1, 0)_{1\leqslant j\leqslant\mu}, (\eta^{-j}e_2, 0)_{1\leqslant j\leqslant\mu}, (0, \eta^{-j}e_1)_{1\leqslant j\leqslant\mu}, (0, \eta^{-j}e_2)_{1\leqslant j\leqslant\mu}\} \quad (3.2)$$

with $\{e_1, e_2\}$ being a basis in $\mathbb{C}^2$. $H^1(\mathscr{U}\times\mathscr{L}, \mathscr{K}(-\mu-1))$ can be regarded as a subspace in $H^1(\mathscr{U}\times\mathscr{L}, \mathscr{O}(-\mu-1)^4)$. An element $(f, g)\in H^1(\mathscr{U}\times\mathscr{L}, \mathscr{O}(-\mu-1)^4)$ belongs to $H^1(\mathscr{U}\times\mathscr{L}, \mathscr{K}(-\mu-1))$ if and only if

$$\xi^\nu f(\xi, \eta)+\varphi(\eta)G(\xi, \eta)g(\xi, \eta)=0 .$$

Since $\det\varphi(\eta)=\mu$ and the expansion of $G(\xi, \eta)$ in the variable η at the point $\eta=\infty$ has the form

$$G(\xi, \eta)=\mathbf{1}+\xi^{-\nu}[\eta^{-1}G_1(\xi)+\eta^{-2}G_2(\xi)+\cdots] , \quad (3.3)$$

the following is true:

(i) A section (f, g) of $\mathscr{K}(-\mu-1)$ defined on $\mathscr{W}_1\cap\mathscr{W}_2$ can be extended on $\mathscr{W}_1$ if and only if $g(\xi, \eta)$ does not contain powers of the variable η greater than $-2\mu-1$.

(ii) A section $(-\varphi(\eta)^2u(\xi, \eta), \xi^\nu\varphi(\eta)G^{-1}(\xi, \eta)u(\xi, \eta))$ of $\mathscr{K}(-\mu-1)$ defined on $\mathscr{W}_1\cap\mathscr{W}_2$ with $u(\xi, \eta)$ not containing negative powers of the variable η can be extended on $\mathscr{W}_2$.

It follows that the vectors

$$\{(-\varphi(\eta)\eta^{\mu-j}e_k, \xi^\nu\eta^{\mu-j}G^{-1}(\xi, \eta)e_k)_{1\leqslant j\leqslant 3\mu,\ k=1,2}\} \quad (3.4)$$

span $H^1(\mathscr{U}\times\mathscr{L}, \mathscr{K}(-\mu-1))$ over the ring $\mathscr{O}(\mathscr{U})$. Using the basis (3.2) and the generators (3.4), one can replace the mapping κ by a $4\mu\times 6\mu$ matrix with entries from the ring $\mathscr{O}=\mathscr{O}(\mathscr{U})$:

$$\begin{pmatrix} \mathbf{0} & -\boldsymbol{T}_1 & -\boldsymbol{T}_2 \\ \Gamma_1 & \Gamma_2 & \mathbf{0} \end{pmatrix}, \quad (3.5)$$

where

$$\Gamma_1=\begin{pmatrix} G^0_\mu & G^0_{\mu-1} & \dots & G^0_1 \\ G^0_{\mu+1} & G^0_\mu & \dots & G^0_2 \\ \vdots & \vdots & & \vdots \\ G^0_{2\mu-1} & G^0_{2\mu-2} & \dots & G^0_\mu \end{pmatrix},$$

$$\Gamma_2=\begin{pmatrix} \xi^\nu\mathbf{1} & \mathbf{0} & \dots & \mathbf{0} \\ G^0_1 & \xi^\nu\mathbf{1} & \dots & \mathbf{0} \\ \vdots & \vdots & & \vdots \\ G^0_{\mu-1} & G^0_{\mu-2} & \dots & \xi^\nu\mathbf{1} \end{pmatrix}.$$

Note that $GG^{-1}=\mathbf{1}$ implies that in order to get Γ_2^{-1} it is enough to replace in Γ_2 the blocks G^0_j by G_j and then multiply the resulting matrix by the term $\xi^{-2\nu}$. The matrices $\boldsymbol{T}_1$, $\boldsymbol{T}_2$ are constant and depend only on the coefficients τ_j of the polynomial $\varphi(\eta)$. $\boldsymbol{T}_1$ is upper triangular, $\boldsymbol{T}_2$ is lower triangular and regular. Consequently, the matrix (3.5) can be further reduced to the $2\mu\times 4\mu$ matrix

$$\boldsymbol{H}(\xi)=(\Gamma_1(\xi),\Gamma_2(\xi))\,.$$

The multiplicity $\sigma(0)$ can then be computed in the following way. It holds that $\dim \operatorname{Ker}(\boldsymbol{H})=2\mu$. Let $\boldsymbol{P}(\xi)$ be a $4\mu\times 2\mu$ matrix with entries from the ring $\mathcal{O}$ and such that $\operatorname{Ker}(\boldsymbol{H})\oplus\operatorname{Ran}(\boldsymbol{P})=\mathcal{O}^{4\mu}$. Then the holomorphic function $\det(\boldsymbol{H}(\xi)\boldsymbol{P}(\xi))$ has a zero in the point $\xi=0$ and the order of this zero is equal to $\sigma(0)$.

Proposition 3.2. *Assume that $\nu>0$ and condition (3.1) is satisfied in all the critical points $(\xi,\eta)=(0,y_j)$, $1\leqslant j\leqslant c$, and put*

$$\Gamma_2^{-1}(\xi)\Gamma_1(\xi)=\xi^{-2\nu}\sum_{j=0}^{\infty}\xi^j Z_j\,,\quad Z_j\in\mathbb{C}^{2\mu,2\mu}\,.$$

Then

$$\sigma(0)=2\mu\nu-\operatorname{rank}\mathfrak{Z}\,,$$

where

$$\mathfrak{Z}=\begin{pmatrix} Z_0 & \mathbf{0} & \cdots & \mathbf{0}\\ Z_1 & Z_0 & \cdots & \mathbf{0}\\ \vdots & \vdots & & \vdots\\ Z_{2\nu-1} & Z_{2\nu-2} & \cdots & Z_0\end{pmatrix}\in\mathbb{C}^{4\mu\nu,4\mu\nu}\,.$$

Proof. The first task is to find a basis in $\operatorname{Ker}\boldsymbol{H}$ over the ring $\mathcal{O}$. Clearly, $(f,g)\in\mathcal{O}^{4\mu}=\mathcal{O}^{2\mu}\oplus\mathcal{O}^{2\mu}$ belongs to $\operatorname{Ker}\boldsymbol{H}$ if and only if

$$g(\xi)=-\Gamma_2^{-1}(\xi)\Gamma_1(\xi)f(\xi)\,.$$

Since $g(\xi)$ has no pole in the point $\xi=0$, one gets a condition on $f(\xi)=f_0+\xi f_1+\xi^2 f_2+\cdots$, viz. $\mathfrak{Z}\varphi=0$, where $\varphi=(f_0,f_1,\ldots,f_{2\nu-1})\in\mathbb{C}^{4\mu\nu}$. When investigating the kernel of $\mathfrak{Z}$ one can reproduce verbatim the analysis given in the proof of lemma 2.2. It is enough to put $\mathfrak{k}=\mathbb{C}$ and replace ν by 2ν and r by 2μ. Then the proof can be followed up to the final part beginning with "Summarizing, ...". The only exception is that now we put

$$F_{jk}(\xi)=\sum_{m=0}^{2\nu-1}\xi^m\pi_m(\varphi_{jk})\,,$$

and besides

$$S_j(\xi)=\xi^{2\nu}g_j\,.$$

We claim that the vectors

$$\{(F_{jk}(\xi), -\Gamma_2^{-1}(\xi)\Gamma_1(\xi)F_{jk}(\xi))_{\substack{0\le j\le 2\nu-1\\ 1\le k\le l_j}}, \quad (S_j(\xi), -\Gamma_2^{-1}(\xi)\Gamma_1(\xi)S_j(\xi))_{1\le j\le d}\} \tag{3.6}$$

form a basis in $\mathrm{Ker}(\boldsymbol{H}(\xi))$ over the ring $\mathcal{O}$. To show this it is enough to set $\xi=0$ in (3.6) and observe that the vectors

$$\{(h_{jk})_{1\le j\le 2\nu-1, 1\le k\le l_j}, (Z_0 g_j)_{1\le j\le d}\}, \tag{3.7}$$

where we have put

$$\begin{aligned} h_{jk} &:= \Gamma_2^{-1}(\xi)\Gamma_1(\xi)F_{jk}(\xi)|_{\xi=0} \\ &= Z_{2\nu-j}f_{jk} + Z_{2\nu-j-1}\pi_{j+1}(\varphi_{jk}) + \cdots + Z_1\pi_{2\nu-1}(\varphi_{jk}), \end{aligned}$$

are independent in $\mathbb{C}^{2\mu}$. Now, choose l_0 vectors $\{u_1, ..., u_{l_0}\}$ completing (3.7) to a basis in $\mathbb{C}^{2\mu}$ and put

$$w_{jk}(\xi) = f_{jk} + \xi\pi_{j+1}(\varphi_{jk}) + \cdots + \xi^{2\nu-j-1}\pi_{2\nu-1}(\varphi_{jk}).$$

Let $\boldsymbol{P}(\xi) \in \mathcal{O}^{4\mu,2\mu}$ be the matrix

$$\boldsymbol{P}(\xi) = \left(\begin{pmatrix} w_{jk} \\ 0 \end{pmatrix}_{\substack{1\le j\le 2\nu-1\\ 1\le k\le l_j}} \quad \begin{pmatrix} g_j \\ 0 \end{pmatrix}_{1\le j\le d} \quad \begin{pmatrix} 0 \\ u_j \end{pmatrix}_{1\le j\le l_0}\right).$$

Then $\mathrm{Ran}(\boldsymbol{P}) \oplus \mathrm{Ker}(\boldsymbol{H}) = \mathbb{C}^{2\mu}$ and $\det(\boldsymbol{H}(\xi)\boldsymbol{P}(\xi)) =$

$$\det\left((\xi^\nu\Gamma_2^{-1}\Gamma_1 w_{jk})_{\substack{1\le j\le 2\nu-1\\ 1\le k\le l_0}} \quad (\xi^\nu\Gamma_2^{-1}\Gamma_1 g_j)_{1\le j\le d} \quad (\xi^\nu u_j)_{1\le j\le l_0}\right).$$

Since

$$\xi^\nu\Gamma_2^{-1}\Gamma_1 w_{jk} = \xi^{\nu-j}h_{jk} + \cdots, \qquad \xi^\nu\Gamma_2^{-1}\Gamma_1 g_j = \xi^{-\nu}Z_0 g_j + \cdots,$$
$$(\wedge h_{jk}) \wedge (\wedge Z_0 g_j) \wedge (\wedge u_j) \neq 0,$$

the lowest power of the variable ξ appearing in $\det(\boldsymbol{H}(\xi)\boldsymbol{P}(\xi))$ with a nonzero coefficient is equal to

$$\begin{aligned} &(\nu-1)l_1 + (\nu-2)l_2 + \cdots + (-\nu+1)l_{2\nu-1} - \nu r + \nu l_0 \\ &= 2\nu l_0 + (2\nu-1)l_1 + \cdots + l_{2\nu-1} - \nu(l_0 + \cdots + l_{2\nu-1} + r) \\ &= \dim \mathrm{Ker}(\mathfrak{Z}) - 2\mu\nu = 2\mu\nu - \mathrm{rank}(\mathfrak{Z}). \end{aligned}$$

This completes the proof. □

4. Derivation of the algebraic equations

In this section we again use the coordinates $\xi = z_1/z_0$, $\eta = z_2/z_0$. The aim is to decide for which values of the parameters s_j, t_j, R_{jk} the following equality holds:

$$c_2(F_G)=c\,, \tag{4.1}$$

provided we are given a matrix-valued function $G(\xi,\eta)$ on $\mathbb{P}^2$. We still assume that $c\geqslant 2$. It turns out that the equality (4.1) leads to a system of algebraic equations of which the derivation is rather straightforward with the help of proposition 3.2.

One can consider only the generic case when the polynomials

$$p_0(\xi)=\xi^c+\sum s_j\xi^{c-j}\,,\qquad p_\infty(\eta)=\eta^c+\sum t_j\eta^{c-j}$$

have no multiple roots. Then $G(\xi,\eta)$ can be written in the form

$$G(\xi,\eta)=\mathbf{1}+\sum_{j,k=1}^{c}\frac{1}{(\xi-x_j)(\eta-y_k)}X_{jk}\,, \tag{4.2}$$

where x_j, $y_j\in\mathbb{C}$, $X_{jk}\in\mathbb{C}^{2,2}$ and $x_1,\ldots,x_c$ are mutually different; the same is true for $y_1,\ldots,y_c$. We shall restrict ourselves to the case when $\operatorname{tr}X_{jk}\neq 0$ for all j, k. This condition is again generic since it can be rewritten as an algebraic one. Actually, $\operatorname{tr}X_{jk}=0$ for some j, k if and only if

$$\prod_{j=1}^{c}\prod_{k=1}^{c}\operatorname{tr}\mathscr{R}(x_j,y_k)=0\,, \tag{4.3}$$

and the left hand side in (4.3) is a symmetric polynomial in the variables $x_1,\ldots,x_c$ and $y_1,\ldots,y_c$ and hence it can be rewritten as a polynomial in the variables $s_1,\ldots,s_c$ and $t_1,\ldots,t_c$.

We know that $c_2(F)$ is equal to the sum of the multiplicities $\sigma(x_j)$ of the jumping lines $\xi-x_j=0$, $j=1,\ldots,c$. Since $\sigma(x_j)\geqslant 1$, the equality $c_2=c$ is satisfied if and only if $\sigma(x_j)=1$ for all j. We shall compute $\sigma(x_1)$; $\sigma(x_2),\ldots,\sigma(x_c)$ can be computed analogously. Condition (3.1) is clearly satisfied and so one can apply proposition 3.2:

$$\sigma(x_j)=2c-\operatorname{rank}\begin{pmatrix}Z_0 & \mathbf{0}\\ Z_1 & Z_0\end{pmatrix}.$$

Let us write

$$X_j=X_{1j}\,,\qquad Q_j=\sum_{k=2}^{c}\frac{1}{x_k-x_1}X_{kj}\,.$$

Then the unimodularity condition $\det G(\xi,\eta)=1$ implies

$$\det X_j=0\,,\qquad \sum_{\substack{n=1\\ n\neq j}}^{c}\frac{1}{y_n-y_j}\operatorname{tr}(X_jX_n^0)=0\,,\qquad \operatorname{tr}(X_jQ_j^0)=0\,,$$

$$\operatorname{tr}X_j+\sum_{\substack{n=1\\ n\neq j}}^{c}\frac{1}{y_n-y_j}\operatorname{tr}(X_jQ_n^0+X_nQ_j^0)=0\,, \tag{4.4}$$

for all j. The two columns of X_j are dependent but, according to the assumption $\mathrm{tr}\, X_j \neq 0$, the matrix X_j is nonzero.

It is easy to find that [recall (3.3)]

$$G_s(\xi) = \sum_{j,k=1}^{c} \frac{\xi - x_1}{\xi - x_j} y_k^{s-1} X_{jk} .$$

Now we can state the explicit formulae for the matrices Z_0, Z_1. But it appears to be more convenient to consider the transformed matrices $Z_0' = U^{-1} Z_0 V^{-1}$, $Z_1' = U^{-1} Z_1 V^{-1}$, where $U = (U_{jk}) \otimes \mathbf{1}$, $V = (V_{jk}) \otimes \mathbf{1}$ and $U_{jk} = y_k^{j-1}$, $V_{jk} = y_j^{c-k}$. After some straightforward computations one arrives at the following expressions. Let us split the matrices Z_0', Z_1' into $2c \times 2$ blocks:

$$Z_0' = (\mathfrak{F}_1 \quad \mathfrak{F}_2 \quad \cdots \quad \mathfrak{F}_c) , \qquad Z_1' = (\mathfrak{G}_1 \quad \mathfrak{G}_2 \quad \cdots \quad \mathfrak{G}_c) .$$

Then

$$\mathfrak{F}_j = \begin{pmatrix} \dfrac{1}{y_1 - y_j} X_1 X_j^0 \\ \dfrac{1}{y_2 - y_j} X_2 X_j^0 \\ \vdots \\ -\sum\limits_{\substack{k=1 \\ k \neq j}}^{c} \dfrac{1}{y_k - y_j} X_k X_j^0 \\ \vdots \\ \dfrac{1}{y_c - y_j} X_c X_j^0 \end{pmatrix} , \qquad \mathfrak{G}_j = \begin{pmatrix} -\dfrac{1}{y_1 - y_j} (X_1 Q_j^0 + Q_1 X_j^0) \\ -\dfrac{1}{y_2 - y_j} (X_2 Q_j^0 + Q_2 X_j^0) \\ \vdots \\ X_j^0 + \sum\limits_{\substack{k=1 \\ k \neq j}}^{c} \dfrac{1}{y_k - y_j} (X_k Q_j^0 + Q_k X_j^0) \\ \vdots \\ -\dfrac{1}{y_c - y_j} (X_c Q_j^0 + Q_c X_j^0) \end{pmatrix} .$$

Lemma 4.1.

(i) $\sigma(x_1) = -2c + \dim \mathrm{Ker}\, \mathfrak{Z}' \geq -c + \dim \mathrm{Ker}\, Z_0'$, *where we have put*

$$\mathfrak{Z}' = \begin{pmatrix} Z_0' & \mathbf{0} \\ Z_1' & Z_0' \end{pmatrix} .$$

(ii) $\dim \mathrm{Ker}\, Z_0' \geq c + 1$ *and hence* $\sigma(x_1) \geq 1$ *(as it should be).*

(iii) Provided $X_k^0 X_j \neq \mathbf{0}$ *for some two indices* j, k *(then necessarily* $j \neq k$*), we have* $\dim \mathrm{Ker}\, Z_0' \geq c + 2$ *and hence* $\sigma(x_1) \geq 2$.

Proof.

(i) Equations (4.4) imply $\mathfrak{G}_j X_j + \mathfrak{F}_j Q_j = \mathbf{0}$. It follows that $\dim \mathrm{Ker}\, \mathfrak{Z}' \geq c + \dim \mathrm{Ker}\, Z_0'$.

(ii) Equations (4.4) imply that $\mathfrak{F}_1 + \mathfrak{F}_2 + \cdots + \mathfrak{F}_c = 0$ and, moreover, $\mathfrak{F}_j X_j = 0$. Put

$$Z_0''=(\mathfrak{Y}_1 \quad \mathfrak{Y}_2 \quad \dots \quad \mathfrak{Y}_{c-1})\,.$$

Then rank $Z_0'=$rank $Z_0''\leqslant 2(c-1)-(c-1)=c-1$. It follows that dim Ker $Z_0'=2c-$rank $Z_0'\geqslant c+1$.

(iii) Suppose, for example, that $X_c^0X_1\neq 0$. Then

$$\mathfrak{Y}_1X_c+\mathfrak{Y}_2X_c+\dots+\mathfrak{Y}_{c-1}X_c=(\mathfrak{Y}_1+\mathfrak{Y}_2+\dots+\mathfrak{Y}_c)X_c=0\,.$$

Due to the condition $X_c^0X_1\neq 0$, one has rank$(X_1, X_c)=2$. Consequently, rank $Z_0''\leqslant c-2$ and dim ker $Z_0'\geqslant c+2$. □

Corollary 4.2. *The equality $c_2(F)=c$ can hold only if the necessary condition*

$$X_{jk}^0X_{jl}=\mathbf{0} \quad \textit{for all } j, k, l\,, \tag{4.5}$$

is satisfied.

It remains to state also a sufficient condition for the equality $\sigma(x_1)=1$. But one can readily verify by checking the matrix $\mathfrak{Z}'$ that this is already the generic case. To complete the discussion it is necessary to rewrite condition (4.5) as a system of algebraic equations for the complex variables s_j, t_j, R_{jk}. Condition (4.5) means that

$$\mathscr{R}(x_j, x_k)^0\mathscr{R}(x_j, y_l)=0 \quad \text{for all } j, k, l\,.$$

Hence the polynomial function $\eta\mapsto\mathscr{R}(x_j, \eta)^0\mathscr{R}(x_j, y_l)$ has c different roots but the degree of this polynomial is equal at most to $(c-1)$. So

$$\mathscr{R}(x_j, \eta)^0\mathscr{R}(x_j, y_l)=0 \quad \text{for all } \eta \text{ and for all } j, l\,.$$

Analogously,

$$\mathscr{R}(x_j, \eta)^0\mathscr{R}(x_j, \mu)=0 \quad \text{for all } \eta, \mu \text{ and all } j\,.$$

It follows that

$$\mathscr{R}(\xi, \eta)^0\mathscr{R}(\xi, \mu)=p_0(\xi)\mathscr{P}(\xi, \eta, \mu)\,, \tag{4.6}$$

where $\mathscr{P}(\xi, \eta, \mu)$ is a matrix-valued polynomial and its degree in the variable ξ is equal at most to $(c-2)$,

$$\mathscr{P}(\xi, \eta, \mu)=\sum_{j=0}^{c-2}\mathscr{P}_j(\eta, \mu)\xi^j\,.$$

Write

$$\mathscr{F}_n(\eta, \mu)=\sum_{m=m_1(n)}^{m_2(n)}\sum_{k=1}^{c}\sum_{l=1}^{c}R_{c-m,k}^0R_{c-n+m,l}\eta^{c-k}\mu^{c-l}\,, \tag{4.7}$$

$$m_1(n)=\max(0, n-c+1)\,, \qquad m_2(n)=\min(c-1, n)\,, \quad 0\leqslant n\leqslant 2c-2\,.$$

Then eq. (4.6) is equivalent to

$$\mathscr{F}_n(\eta,\mu)=\sum_{j=j_1(n)}^{j_2(n)} s_j\mathscr{P}_{n-c+j}(\eta,\mu)\,,\quad 1\leqslant n\leqslant 2c-2\,, \tag{4.8}$$

where $j_1(n)=\max(0,\ c-n)$, $j_2(n)=\min(c,\ 2c-2-n)$. The last $(c-1)$ equations in (4.8) can be used to express the polynomials $\mathscr{P}_0(\xi,\eta),\ldots,\mathscr{P}_{c-2}(\xi,\eta)$ with the help of the polynomials $\mathscr{F}_c(\xi,\eta),\ldots,\mathscr{F}_{2c-2}(\xi,\eta)$. Substituting them into the first c equations one gets the desired system of algebraic equations

$$\begin{pmatrix}\mathscr{F}_{c-1}\\ \mathscr{F}_{c-2}\\ \mathscr{F}_{c-3}\\ \vdots\\ \mathscr{F}_0\end{pmatrix}=\begin{pmatrix}s_{c-1} & s_{c-2} & \cdots & s_1\\ s_c & s_{c-1} & \cdots & s_2\\ 0 & s_c & \cdots & s_3\\ \vdots & \vdots & & \vdots\\ 0 & 0 & \cdots & s_c\end{pmatrix}\begin{pmatrix}1 & 0 & \cdots & 0\\ s_1^* & 1 & \cdots & 0\\ s_2^* & s_1^* & \cdots & 0\\ \vdots & \vdots & & \vdots\\ s_{c-2}^* & s_{c-3}^* & \cdots & 1\end{pmatrix}\begin{pmatrix}\mathscr{F}_{2c-2}\\ \mathscr{F}_{2c-3}\\ \mathscr{F}_{2c-4}\\ \vdots\\ \mathscr{F}_c\end{pmatrix}, \tag{4.9}$$

$$s_j^*=(-1)^j\det\begin{pmatrix}s_1 & 1 & 0 & \cdots & 0\\ s_2 & s_1 & 1 & \cdots & 0\\ \vdots & \vdots & \vdots & & \vdots\\ s_j & s_{j-1} & s_{j-2} & \cdots & s_1\end{pmatrix}.$$

Both sides in (4.9) are polynomials in the variables η, μ and the equality means that the corresponding coefficients are equal. The system (4.9) is additional to the unimodularity condition $\det G(\xi,\eta)=1$. This condition can be rewritten as

$$\not{p}_0(\xi)\not{p}_\infty(\eta)\ \mathrm{tr}\ \mathscr{R}(\xi,\eta)+\det\ \mathscr{R}(\xi,\eta)=0\,. \tag{4.10}$$

This equality again means that the coefficients of the left hand polynomial vanish. Let us now summarize the result.

Theorem 4.3. *Put $\mathscr{U}=\mathbb{C}^{2c(2c+1)}\setminus\mathscr{E}$, where $\mathscr{E}$ is the algebraic set determined by the equality (4.3) and by the condition that the discriminant of at least one of the polynomials $\not{p}_0(\xi)$, $\not{p}_\infty(\eta)$ vanishes. Then eqs. (4.9) and (4.10) determine an irreducible algebraic set $\mathscr{A}(2,c)$ in the open set $\mathscr{U}$, $\dim\mathscr{A}(2,c)=4c$, and $\mathscr{O}M(2,c)\cap\mathscr{U}\subset\mathscr{A}(2,c)$.*

5. Concluding remark

The concept of the canonical transition function can be developed also for framed holomorphic bundles on higher-dimensional projective spaces (cf. ref. [3]). But the problem of removability of singularities has not yet been solved for dimensions higher than two. In particular, it would be interesting to have a solution to this problem in the three-dimensional case as it is closely related to instantons. It is reasonable to expect that the ability to treat singularities on $\mathbb{P}^3$

could provide a deeper insight into Donaldson's restriction mapping, similar to that already achieved for the two-instanton case (cf. the appendix).

Consider the standard situation on $\mathbb{P}^n$, $n \geq 3$. $\mathscr{L}_0 \subset \mathbb{P}^n$ is a fixed line and P_0, $P_\infty \in \mathscr{L}_0$ are two fixed distinct points. Suppose G is a rational matrix-valued function on $\mathbb{P}^n$ of some appropriate form. Choose a projective subspace $\mathbb{P}^{n-2}$ in $\mathbb{P}^n$ not intersecting the line $\mathscr{L}_0$. To each point Q from $\mathbb{P}^{n-2}$ there corresponds a plane $\mathscr{P}_Q = \overline{Q\mathscr{L}_0}$. Then the restriction $G_Q = G|\mathscr{P}_Q$ determines a framed holomorphic bundle F_Q on $\mathscr{P}_Q$. Let $c_2(Q)$ designate the value of the second Chern class of this bundle. A necessary condition for the singularities on $\mathbb{P}^n$ to be removable is

$$c_2(Q) = \text{constant on } \mathbb{P}^{n-2}\,.$$

Let us conjecture that this condition is also sufficient.

Appendix. Donaldson's restriction for two instantons

As was mentioned in the introduction, $G(\xi, \eta)$ regarded as a transition function patches together two open sets $\mathscr{U}_0$, $\mathscr{U}_\infty$ in $\mathbb{P}^2$. But one can change the order of these sets, i.e., replace the points P_0, $P_\infty \in \mathscr{L}_0$ by one another. After this transformation, $G(\xi, \eta)$ is replaced by $G(\eta, \xi)^{-1}$ and for the parameters this means that x_j is replaced by y_j and y_k by x_k and X_{jk} by X^0_{kj} in formula (4.2). Clearly, this transformation does not influence the topological charge. Consequently, one gets, in addition to (4.5), another necessary condition for the equality $c_2(F) = c$. Namely,

$$X_{kj} X^0_{lj} = \mathbf{0} \quad \text{for all } j, k, l\,.$$

These conditions suggest a parameterization of the complex manifold $M(2, c)$. The matrices X_{jk} should have [almost everywhere on $M(2, c)$] the form

$$X_{jk} = \epsilon_{jk} \begin{pmatrix} u_j v_k & u_j \\ v_k & 1 \end{pmatrix}.$$

So the points from $M(2, c)$ are determined by the $4c$ parameters $x_1, \ldots, x_c$; $y_1, \ldots, y_c$; $u_1, \ldots, u_c$; $v_1, \ldots, v_c$ and the parameterization is unique up to the numbering of the roots $x_1, \ldots, x_c$ and $y_1, \ldots, y_c$. The unimodularity implies that the c^2 complex parameters ϵ_{jk} should be obtained as a (nontrivial) solution of the following system of c^2 quadratic equations:

$$(1 + y_j v_k)\epsilon_{jk} + \sum_{\substack{m=1 \\ m \neq j}}^{c} \sum_{\substack{n=1 \\ n \neq k}}^{c} \frac{(u_m - u_j)(v_n - v_k)}{(x_m - x_j)(y_n - y_k)} (\epsilon_{jk}\epsilon_{mn} - \epsilon_{jn}\epsilon_{mk}) = 0\,,$$

for $j, k = 1, \ldots, c$. Since

$$\sum_{j=1}^{c} (1+u_j v_k)\epsilon_{jk} = \sum_{j=1}^{c} (1+u_k v_j)\epsilon_{kj} = 0\,,$$

the number of quadratic equations can be reduced to $(c-1)^2$.

The comparatively small number of parameters enables one to treat the two-instanton case in more detail. Particularly, one is able to solve the above quadratic equations and write down the parameterization of $M(2,2)$ explicitly. It is interesting that this parameterization enables one to find the inversion of Donaldson's one-to-one mapping $M(2,2)\to \mathcal{O}M(2,2)$ in explicit form as well. The formulae stated below were found by trial and error.

The manifold $\mathcal{O}M(2,c)$ is parameterized with the help of eight independent complex variables $x_1, x_2, y_1, y_2, u_1, u_2, v_1, v_2$ satisfying the restrictions

$$x_1 \neq x_2\,,\quad y_1 \neq y_2\,,\quad u_1 \neq u_2\,,\quad v_1 \neq v_2\,,$$
$$1+u_j v_k \neq 0 \quad \text{for all } j, k\,.$$

The solution of the quadratic equations is then given by

$$\epsilon_{jk} = (-1)^{j+k} \frac{\epsilon}{1+u_j v_k}\,,$$

$$\epsilon = (x_2-x_1)(y_2-y_1) \times \frac{(1+u_1v_1)(1+u_1v_2)(1+u_2v_1)(1+u_2v_2)}{(u_2-u_1)^2(v_2-v_1)^2}\,.$$

This parameterization is unique up to the numbering of the roots x_1, x_2 and y_1, y_2. So the parameterized set is a four-fold covering of an open dense subset in $\mathcal{O}M$. The group $\mathbb{Z}_2\times\mathbb{Z}_2$ acts transitively on the fibres of this covering.

A two-instanton solution in the ADHM construction (cf. ref. [5]) is determined by a quadruple of matrices $\alpha_1, \alpha_2, a, b \in \mathbb{C}^{2,2}$ which satisfy

$$ba + [\alpha_1, \alpha_2] = 0\,,$$
$$[\alpha_1, \alpha_1^\dagger] + [\alpha_2, \alpha_2^\dagger] + bb^\dagger - a^\dagger a = 0\,.$$

The CTF $G(\xi,\eta)$ is related to the ADHM data $(\alpha_1, \alpha_2, a, b)$ by the identity (cf. ref. [3])

$$G(\xi,\eta) = \mathbf{1} + a(\xi\mathbf{1}-\alpha_1)^{-1}(\eta\mathbf{1}+\alpha_2)^{-1}b\,.$$

To find the inversion to Donaldson's mapping means to relate to the eight parameters x_j, y_j, u_j, v_j a quadruple $(\alpha_1, \alpha_2, a, b)$.

First, express the parameters u_1, u_2, v_1, v_2 with the help of other complex parameters $\kappa, \mu, \nu, \varphi, \psi$:

$$u_1=\kappa^2\frac{\sinh(\varphi-\mu)}{\cosh(\varphi+\nu)}, \qquad u_2=\kappa^2\frac{\cosh(\varphi-\mu)}{\sinh(\varphi+\nu)},$$
$$v_1=\kappa^{-2}\frac{\sinh(\psi-\nu)}{\cosh(\psi+\mu)}, \qquad v_2=\kappa^{-2}\frac{\cosh(\psi-\nu)}{\sinh(\psi+\mu)},$$

and put

$$\omega=\left(-(x_2-x_1)(y_2-y_1)\frac{\sinh(2\varphi+2\psi)}{\cosh(\mu+\nu)}\right)^{1/2}.$$

The solution of these equations is not unique. Provided κ, μ_0, ν_0, φ_0, ψ_0 is a solution, then κ, $\mu_0+\tau$, $\nu_0-\tau$, $\varphi_0+\tau$, $\psi_0-\tau$ will do as well. The equation $f(\tau)=0$, where

$$f(\tau)=|x_2-x_1|^2\sinh(4\tau+4\operatorname{Re}\varphi_0)+|y_2-y_1|^2\sinh(4\tau-4\operatorname{Re}\psi_0)$$
$$+|\kappa\omega|^2\sinh(2\tau+2\operatorname{Re}\mu_0)+|\omega/\kappa|^2\sinh(2\tau-2\operatorname{Re}\nu_0)$$

is a real function in one real variable, has the unique solution $\tau_0\in\mathbb{R}$. Put

$$\mu=\mu_0+\tau_0, \quad \nu=\nu_0-\tau_0, \quad \varphi=\varphi_0+\tau_0, \quad \psi=\psi_0-\tau_0.$$

Then the desired explicit expressions for α_1, α_2, a, b are as follows:

$$\alpha_1=x_1\,\mathbf{1}+\frac{x_2-x_1}{2}\begin{pmatrix}1 & -e^{-2\varphi}\\ -e^{2\varphi} & 1\end{pmatrix},$$

$$\alpha_2=-y_1\,\mathbf{1}-\frac{y_2-y_1}{2}\begin{pmatrix}1 & e^{2\psi}\\ e^{-2\psi} & 1\end{pmatrix},$$

$$a=\frac{\omega}{2}\begin{pmatrix}-\kappa e^{\mu} & \kappa e^{-\mu}\\ \kappa^{-1}e^{-\nu} & \kappa^{-1}e^{\nu}\end{pmatrix}, \qquad b=\frac{\omega}{2}\begin{pmatrix}\kappa^{-1}e^{\nu} & -\kappa e^{-\mu}\\ \kappa^{-1}e^{-\nu} & \kappa e^{\mu}\end{pmatrix}.$$

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